Inequality
https://www.linkedin.com/groups/8313943/8313943-6440107986907201540
Let a,b,c be positive real numbers such that
$abc = 1$, prove that

$(a+b+c+3)/4 \geq 1/(a+b) + 1/(b+c) + 1/(c+a)$.

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By replacing (a,b,c) with (x^3,y^3,z^3) we obtain the following equivalent inequality
$\dfrac{x^3+y^3+z^3+3}{4} \geq \sum \dfrac{1}{x^3+y^3}$ for $x,y,z > 0$ and $xyz = 1$ which in homogeneous is

(**1**) $\dfrac{x^3+y^3+z^3+3xyz}{4xyz} \geq \sum \dfrac{xyz}{x^3+y^3}$.

Since $x^3+y^3 \geq x^2y+xy^2 = xy(x+y)$ then $\sum \dfrac{xyz}{x^3+y^3} \leq \sum \dfrac{z}{x+y}$ and for proving inequality (**1**) remains to prove inequality

(**2**) $\dfrac{x^3+y^3+z^3+3xyz}{4xyz} \geq \sum \dfrac{z}{x+y} \Leftrightarrow \dfrac{x^3+y^3+z^3+3xyz}{4xyz} + 3 \geq \sum\left(\dfrac{z}{x+y}+1\right) \Leftrightarrow$
$\dfrac{x^3+y^3+z^3+15xyz}{4xyz} \geq (x+y+z)\sum \dfrac{1}{x+y}$.

Assuming $x+y+z = 1$ (due homogeneity of inequality (**2**)) and denoting
$p := xy+yz+zx, q := xyz > 0$ we obtain $x^3+y^3+z^3+15xyz = 1+18q-3p$,
$(x+y+z)\sum \dfrac{1}{x+y} = \dfrac{1+p}{p-q}$. Then inequality (**2**) becomes
$\dfrac{1+18q-3p}{4q} \geq \dfrac{1+p}{p-q} \Leftrightarrow (1+18q-3p)(p-q) \geq 4q(1+p) \Leftrightarrow$

$p(1-3p) \geq 18q^2+(5-17p)q$.

Noting that $p = xy+yz+zx \leq (x+y+z)^2/3 = 1/3$ and
$q = xyz(x+y+z) \leq (xy+yz+zx)^2/3 = p^2/3$ and taking in account that
for $q \in [0,p^2/3]$ we have $\max(18q^2+(5-17p)q) =$
$\max\left\{0, 18\cdot\left(\dfrac{p^2}{3}\right)^2+(5-17p)\dfrac{p^2}{3}\right\} = \dfrac{1}{3}p^2(1-3p)(5-2p)$
we obtain $p(1-3p)-(10q^2+(5-9p)q) \geq p(1-3p)-\dfrac{1}{3}p^2(1-3p)(5-2p) =$
$\dfrac{1-3p}{3}(3p-p^2(5-2p)) = \dfrac{p(1-p)(1-3p)(3-2p)}{3} \geq 0.$